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Some Remarks on the Almost Sure Central Limit Theorem for Dependent Sequences

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Let $(Z_i)_{i=1}^{\infty}$ be a stationary mean zero Gaussian sequence with covariance at lag j equal to $L(j)/j^{\alpha}$ where $\alpha > 0$ and $L(\cdot)$ is ultimately positive function slowly varying at infinity. We prove that for all α $(Z_i)_{i=1}^{\infty}$ satisfies almost sure Central Limit Theorem. For $\alpha < 1$ the result was demonstrated by a different method in [16] as a partial case of a more general result for a long-range dependent sequence $\left(G(Z_i)\right)_{i=1}^{\infty}$. When the sequence $\left(G(Z_i)\right)_{i=1}^{\infty}$ is short-range dependent i.e. its absolute covariances are summable, we prove that it satisfies almost sure CLT provided a spectral density of the sequence $(Z_i)_{i=1}^{\infty}$ is bounded away from 0 and is logarithmic Lipschitz. Finally, a functional version of this result is proved.

1. Introduction

Let $Z_1, Z_1, \ldots$ be a stationary sequence defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $EZ_1 = 0$ and $EZ_1^2 = 1$. Put $S_n := Z_1 + Z_2 + \ldots + Z_n$ and $\sigma_n^2 := \operatorname{Var} S_n = ES_n^2$. $I_A(\cdot)$ will stand for an indicator function of a set A. The almost sure properties of S_n after logarithmic averaging have become an intensively studied subject in recent research. In particular, the following property is investigated: there exists a $\mathbb{P}$-null set $N \subset \Omega$ such that for all ω belonging to its complement

$$(1.1) \qquad (\log n)^{-1} \sum_{k \leq n} k^{-1} I_A\big(S_k(\omega)/\sigma_k\big) \quad \to \quad (2\pi)^{-1/2} \int_A e^{-u^2/2}\, du$$

for all Borel sets $A \subset \mathbb{R}$ with $\lambda(\partial A) = 0$. Note that average in (1.1) is equivalent to a weighted time average with weights

$$w_{ni} = (\log n)^{-1}\{\log(i+1) - \log i\} \sim (i \log n)^{-1} \quad \text{for} \quad i = 1, \dots, n.$$

For i.i.d. sequence property (1.1) was proved by Brosamler [6] and Schatte (1988) provided $E|Z_1|^{2+\delta}$ exists for some $\delta > 0$ ($\delta = 1$ was assumed in the later paper) and the final result with this condition discarded is due to Lacey and Philipp [17] and Fischer [11]. For independent not necessarily identically distributed random variables a general result was proved by Berkes and Dehling [3]. We also cite in this context Csáki, Földes and Révész [7], Horváth and M. Csörgő [8], Major [19] and Szabłowski [24] among others; for the recent review see [1]. Let us only note that an existence of a distributional limit of S_n/σ_n is not necessary for a.s. convergence of left hand side in (1.1) (cf. e.g. [3] and [2]). Actually, S_n/σ_n may have different limit distributions along different subsequences while left hand side of (1.1) converges almost surely.

The objective of this note is to study property (1.1) for certain classes of dependent sequences $(Z_i)_{i=1}^{\infty}$. For this setup two main methods are available. Lacey and Philipp [17] proved that (1.1) holds provided $(Z_i)_{i=1}^{\infty}$ satisfies almost sure invariance principle with the rate $o(n^{1/2})$ i.e. if there exist a Wiener process $W(\cdot)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$S_n - W(n) = o(n^{1/2}) \quad \text{a.s.} \tag{1.2}$$

Peligrad and Shao [22] showed that given $S_n/\sigma_n \to N(0,1)$ in distribution, (1.1) holds if for any bounded, Lipschitz function $f : \mathbb{R} \mapsto \mathbb{R}$

$$\operatorname{Var}\left(\sum_{k=1}^{n} \frac{1}{k} f\left(\frac{S_k}{\sigma_k}\right)\right) = \mathcal{O}(\log^{2-\varepsilon} n), \tag{1.3}$$

where ε is some positive constant. For relevant results in this area see also [16] and [13]. In this note we study a situation when $(Z_i)_{i=1}^{\infty}$ is a Gaussian sequence with an approxiamately hiperbolic covariance function: $r(i) := \operatorname{Cov}(Z_1, Z_{1+i}) = L(i)i^{-\alpha}$ where $\alpha > 0$ and $L(\cdot)$ is a slowly varying function. We impose no restriction on α apart from its positivity. Although some partial results for a Gaussian case may be obtained using almost sure invariance principle approach (using e.g. a result of Morrow [21]), in order to obtain more complete picture we found it preferable to apply Peligrad–Shao criterion (1.3). Suprisingly at the first sight, it turns out that condition (1.3) is satisfied even in the long-range dependent situation i.e. when $\alpha < 1$ as well for $\alpha = 1$ provided $L(\cdot)$ satisfies certain regularity conditions. Moreover, this methodology may be also employed to study a.s. CLT for Gaussian subordinated process $\left(G(Z_i)\right)_{i=1}^{\infty}$, where Z_i's are Gaussian, when the transformed sequence is short-range dependent i.e. a sum of its absolute covariances is finite. In Theorem 4 we also prove its functional analogue (cf. (2.8)). A counterpart of this result for the long-range dependent case was obtained by Lacey [16] who studied properties of S_n relying on its representation as a multiple Wiener–Itô integral.

2. Main Result

Let $(Z_i)_{i=1}^{\infty}$ be a mean-zero stationary Gaussian process such that $EZ_1^2 = 1$ with a covariance function at a lag j

$$r(j) := E(Z_1 Z_{1+j}) = L(j)/j^{\alpha}, \qquad j = 1, 2, \ldots$$

where $\alpha > 0$ and $L(\cdot)$ is an ultimately positive function slowly varying at infinity.

We prove the following result

Theorem 1. *Assume that the above conditions are satisfied and $\alpha > 0$. Then almost sure CLT (1.1) holds.*

Proof. In the derivations C denotes a generic constant value of which may change from line to line. Obviously, since Z_i's are jointly Gaussian, S_n/σ_n has the standard normal distribution and the Central Limit Theorem for S_n/σ_n trivially holds. Thus in view of Theorem 1 in [22] it is enough to prove that condition (1.3) is valid for some positive ε and any bounded Lipschitz function f. Assume first that $0 < \alpha < 1$. We have

$$\operatorname{Var}\left(\sum_{i=1}^{n} \frac{1}{i} f\left(\frac{S_i}{\sigma_i}\right)\right) = I_1 + I_2$$

$$:= \sum_{i=1}^{n} \frac{1}{i^2} \operatorname{Var}\left(f\left(\frac{S_i}{\sigma_i}\right)\right) + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \operatorname{Cov}\left(f\left(\frac{S_i}{\sigma_i}\right), f\left(\frac{S_j}{\sigma_j}\right)\right).$$

Since f is bounded we obviously have $I_1 \leq C$. Moreover, following the proof of Lemma 1 in [22] we write

$$\text{(2.1)}\quad \operatorname{Cov}\left(f\left(\frac{S_i}{\sigma_i}\right), f\left(\frac{S_j}{\sigma_j}\right)\right) = \operatorname{Cov}\left(f\left(\frac{S_i}{\sigma_i}\right), f\left(\frac{S_{j+2i}-S_{2i}}{\sigma_j}\right)\right)$$
$$+Ef\left(\frac{S_i}{\sigma_i}\right)\left(f\left(\frac{S_j}{\sigma_j}\right) - f\left(\frac{S_{j+2i}-S_{2i}}{\sigma_j}\right)\right)$$
$$-Ef\left(\frac{S_i}{\sigma_i}\right)E\left(f\left(\frac{S_j}{\sigma_j}\right) - f\left(\frac{S_{j+2i}-S_{2i}}{\sigma_j}\right)\right).$$

The last term is zero in view of stationarity of $(Z_i)_{i=1}^{\infty}$. Two times application of Karamata's theorem yields $\sigma_n^2 \sim D_\alpha L(n) n^{2-\alpha}$, where $D_\alpha = 2/\big((1-\alpha)(2-\alpha)\big)$. This implies that the conditions

$$\text{(2.2)}\quad \sigma_{2i} \leq 2\sigma_i \quad \text{and} \quad \sum_{j=i+1}^{\infty} (j\sigma_j)^{-1} \leq 4/\sigma_i$$

are fulfilled for large i. But as an examination of the proof of Lemma 1 in [22] indicates only these two conditions are used to prove that a sum of penultimate terms in (2.1) over i, j such that $1 \leq i < j \leq n$ is smaller than

$$C\left(\sum_{j=1}^{\infty} (j\sigma_j)^{-1} + \log n\right) = \mathcal{O}(\log n).$$

Thus the main problem remains to deal with a sum of first terms in equality (2.1). Obviously, without loss of generality we may assume that f is such that $\operatorname{Var} f(Z_1) = 1$. Since f is bounded, $f \in \mathcal{L}^2(\mathbb{R}, \phi)$, where ϕ denotes the standard normal density weight. Then Gebelein–Lancaster inequality ([18]) applied to a Gaussian vector $(S_i/\sigma_i, (S_{j+2i} - S_{2i})/\sigma_j)$ yields

$$\text{(2.3)}\quad \left|\operatorname{Cov}\left(f\left(\frac{S_i}{\sigma_i}\right), f\left(\frac{S_{j+2i}-S_{2i}}{\sigma_j}\right)\right)\right| \leq \left|\operatorname{Cov}\left(\frac{S_i}{\sigma_i}, \frac{S_{j+2i}-S_{2i}}{\sigma_j}\right)\right|.$$

Moreover,

$$\operatorname{Cov}(S_i, S_{2i+j} - S_{2i}) = \sum_{\substack{2i+1 \leq l \leq j+2i \\ 1 \leq k \leq i}} r(l-k).$$

Note that without loss of generality we may assume that $r(i) \geq 0$ for all $i \in \mathbb{N}$. Indeed, for any fixed $i_0 \in \mathbb{N}$

$$\sum_{j=1}^{n} \left|\operatorname{Cov}\left(\frac{1}{i_0} f\left(\frac{S_{i_0}}{\sigma_{i_0}}\right), \frac{1}{j} f\left(\frac{S_j}{\sigma_j}\right)\right)\right| \leq C \sum_{j=1}^{n} \frac{1}{j} = \mathcal{O}(\log n)$$

in view of boundedness of f. Assuming this from now on we have

$$(2.4)\quad \big|\operatorname{Cov}(S_i, S_{2i+j} - S_{2i})\big| \le i \sum_{s=i+1}^{j+2i-1} r(s) = \mathcal{O}\big(iL(j+2i)(j+2i)^{1-\alpha}\big),$$

where the last relation is a consequence of Karamata's theorem (cf. [23, Theorem 0.6]) and a fact that $0 < \alpha < 1$. Thus (2.4) implies

$$(2.5)\qquad \sum_{1\le i<j\le n} \frac{1}{ij}\left|\operatorname{Cov}\left(\frac{S_i}{\sigma_i}, \frac{S_{2i+j}-S_{2i}}{\sigma_j}\right)\right|$$

$$\le C \sum_{1\le i<j\le n} \frac{i}{(ij)^{2-\alpha/2}} \frac{L(j+2i)}{L(j)^{1/2}L(i)^{1/2}} (j+2i)^{1-\alpha}$$

$$= C\sum_{i=1}^{n-1} \frac{1}{i^{1-\alpha/2}L(i)^{1/2}} \sum_{j>i} \frac{L(j+2i)}{L(j)^{1/2}} \left(1+\frac{2i}{j}\right)^{1-\alpha} \frac{1}{j^{1+\alpha/2}}.$$

Since $(1+2i/j)^{1-\alpha} < 3$ for $i < j$, the inner sum is bounded by

$$C\sum_{j=i}^{\infty} \bar{L}(j) j^{-1-\alpha/2},$$

where, for a given i, $\bar{L}(j) := L(j+2i)/L(j)^{1/2}$ is a slowly varying function. Since from Karamata's theorem it follows that

$$\lim_{i\to\infty} \frac{\sum_{j=i}^{\infty} \bar{L}(j) j^{-1-\alpha/2}}{\frac{2}{\alpha}\bar{L}(i) i^{-\alpha/2}} \to 1$$

we have that (2.5) is smaller than $C\sum_{i=1}^{n} \bar{L}(i)L^{-1/2}(i)1/i$ and since

$$\bar{L}(i)L^{-1/2}(i) = L(3i)/L(i) = \mathcal{O}(1)$$

in view of the definition of slow variation, (2.5) is $\mathcal{O}(\log n)$ and thus condition (1.3) is satisfied with $\varepsilon = 1$.

Consider now the case $\alpha > 1$. In this assumption $\sigma_n^2/n \to r(0) + \sum_{i=1}^{\infty} r(i) > 0$ and conditions (2.2) are trivially satisfied. Moreover,

$$i\sum_{s=i+1}^{j+2i-1} r(s) = i\left(\sum_{s=i+1}^{\infty} r(s) - \sum_{s=j+2i}^{\infty} r(s)\right) = \mathcal{O}\big(iL(i)i^{1-\alpha}\big)$$

in view of Karamata's theorem. Since $\sigma_i^2 \geq Ci$ in this case, reasoning as before we have

$$\sum_{1\leq i<j\leq n} \frac{1}{ij}\left|\mathrm{Cov}\left(\frac{S_i}{\sigma_i}, \frac{S_{2i+j}-S_{2i}}{\sigma_j}\right)\right| = \mathcal{O}\bigg(\sum_{1\leq i<j\leq n} \frac{1}{ij}\frac{i}{(ij)^{1/2}}L(i)i^{1-\alpha}\bigg)$$
$$= \mathcal{O}\bigg(\sum_{i=1}^{n} L(i)i^{1/2-\alpha} \sum_{n\geq j>i\geq 1} \frac{1}{j^{3/2}}\bigg) = \mathcal{O}\bigg(\sum_{i=1}^{n} \frac{L(i)}{i^{\alpha}}\bigg) = \mathcal{O}(1),$$

since $L(i) \leq Ci^{\varepsilon}$ for arbitrary $\varepsilon > 0$ and $\alpha > 1$. Thus for $\alpha > 1$ (1.3) holds with $\varepsilon = 2$.

Consider now the case $\alpha = 1$. Note that, reasoning as before, we may restrict ourselves to a situation when $L(i)$ is positive for all $i \in \mathbb{N}$. Observe first that since $L(sn)/L(n)$ tends to 1 when $n \to \infty$ uniformly in $s \in [a,b]$, where $0 < a < b < \infty$, ([23, Proposition 0.5]) thus for any $t > 0$ we have

$$\sum_{i=1}^{n} r(i) = \sum_{i=1}^{n} \frac{L(i)}{i} \geq \sum_{i=[nt]}^{n} \frac{L(i)}{i} \sim L(n) \sum_{i=[nt]}^{n} \frac{1}{i} \sim L(n)\log 1/t,$$

whereas in view of Karamata's theorem

$$\sum_{i=1}^{n} ir(i) = \sum_{i=1}^{n} L(i) = \mathcal{O}\big(nL(n)\big).$$

Since $t > 0$ is arbitrary, it follows that $\sum_{i=1}^{n} ir(i) = o\big(n\sum_{i=1}^{n} r(i)\big)$. Thus

$$\sigma_n^2 = nr(0) + 2\sum_{i=1}^{n-1}(n-i)r(i) \sim n\bigg(1 + 2\sum_{i=1}^{n-1} r(i)\bigg) =: n\tilde{L}(n)$$

and $\tilde{L}(n)$ is a slowly varying function implied by

$$\sum_{i=n+1}^{[nt]} L(i)/i = o\bigg(\sum_{i=1}^{n} L(i)/i\bigg)$$

for any $t > 1$. Thus conditions (2.2) are satisfied. We proceed analogously to the proof for $\alpha < 1$ using (2.1), (2.3) and the first inequality in (2.4). But now using definition of $\tilde{L}(\cdot)$ we have

$$i\sum_{s=i+1}^{j+2i-1} r(s) = i\big(\tilde{L}(j+2i) - \tilde{L}(i+1)\big)/2.$$

Thus in view of $\sigma_i^2 \sim i\tilde{L}(i)$

$$(2.6) \qquad \sum_{1\le i<j\le n} \frac{1}{ij}\left|\operatorname{Cov}\left(\frac{S_i}{\sigma_i}, \frac{S_{2i+j}-S_{2j}}{\sigma_j}\right)\right|$$

$$= \mathcal{O}\left(\sum_{i=1}^{n} \frac{1}{i^{1/2}\tilde{L}(i)^{1/2}} \sum_{j\ge i} \frac{\tilde{L}(j+2i)-\tilde{L}(i+1)}{j^{3/2}\tilde{L}(j)^{1/2}}\right)$$

Observe that for a fixed i $L_0(j) := \left(\tilde{L}(j+2i)-\tilde{L}(i+1)\right)/\tilde{L}^{1/2}(j)$ is a slowly varying function at ∞ since it is a ratio of slowly varying functions and $\tilde{L}(\cdot)$ is an increasing function. Thus from Karamata's theorem right hand side of (2.6) is of the order

$$(2.7) \qquad \mathcal{O}\left(\sum_{i=1}^{n} \frac{1}{i^{1/2}\tilde{L}^{1/2}(i)} \frac{\left(\tilde{L}(3i)-\tilde{L}(i+1)\right)}{i^{1/2}\tilde{L}^{1/2}(i)}\right).$$

But using a fact that $\left(\tilde{L}(3i)-\tilde{L}(i+1)\right)/\tilde{L}(i)$ is bounded since $\tilde{L}(\cdot)$ is slowly varying we have that $(2.7) = \mathcal{O}(\log n)$ and (1.3) satisfied with $\varepsilon = 1$. ■

Remark. (a) Theorem 1 for $0 < \alpha < 1$ was proved by Lacey [16] as a partial case of a more general result stating almost sure CLT for Gaussian subordinated sequence $\left(G(Z_i)\right)_{i=1}^{\infty}$ for $G \in \mathcal{L}^2(\mathbb{R}, \phi)$ by a method using Wiener–Itô integral representation of S_n. In Theorem 3 below we prove a counterpart of this result for short-range Gaussian subordinated sequence i.e. such that its absolute covariances are summable.

(b) Observe also that if $L(i) \ge 0$ for *all* i then $(Z_i)_{i=1}^{\infty}$ is associated and then Theorem 1 for $\alpha > 1$ follows from Peligrad and Shao's [22] result stating that a.s. CLT holds for any associated sequence with summable covariances.

It turns out that Theorem 1 for $\alpha > 1$ may be generalized to an arbitrary form of covariance function.

Theorem 2. *Assume that $(Z_i)_{i=1}^{\infty}$ is a stationary mean-zero Gaussian sequence with covariance function satisfying $\sum_{i=0}^{\infty}\left|r(i)\right| < \infty$ and $\sigma^2 = r(0) + 2\sum_{i=1}^{\infty} r(i) \neq 0$. Then a.s. CLT* (1.1) *holds.*

Proof. Observe that in view of Lemma 3, Section 20 in [5] $ES_n^2/n \to \sigma^2 \neq 0$, thus $ES_n^2 \ge Cn$ for sufficiently large n. Reasoning as in proof of Theorem 1 we may assume that the last inequality holds for all n. From

Gebelein–Lancaster inequality it follows that $\left|\operatorname{Cov}\big(f(S_i/\sigma_i), f(S_j/\sigma_j)\big)\right| \le \left|\operatorname{Cov}(S_i/\sigma_i, S_j/\sigma_j)\right|$ for f such that $\operatorname{Var}\big(f(Z_1)\big) = 1$. Moreover,

$$\sum_{1\le i<j\le n} \frac{1}{ij}\left|\operatorname{Cov}\left(\frac{S_i}{\sigma_i}, \frac{S_j}{\sigma_j}\right)\right| \le C\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{j}\frac{\left|\operatorname{Cov}(Z_k, Z_l)\right|}{i^{3/2}j^{3/2}}$$

$$\le C\sum_{k=1}^{n-1}\sum_{l=1}^{n}\sum_{i=k}^{n-1}\sum_{j=l}^{n}\frac{\left|\operatorname{Cov}(Z_k, Z_l)\right|}{i^{3/2}j^{3/2}} \le C\sum_{k=1}^{n-1}\sum_{l=1}^{n}\frac{\left|\operatorname{Cov}(Z_k, Z_l)\right|}{k^{1/2}l^{1/2}}$$

$$\le 2C\sum_{k=1}^{n-1}\sum_{l=k}^{n}\frac{1}{k}\left|\operatorname{Cov}(Z_1, Z_{l-k+1})\right| \le 2C\sum_{i=1}^{\infty}\left|\operatorname{Cov}(Z_1, Z_i)\right|\log n$$

and the result follows from (1.3). ■

Remark. Observe that under more stringent condition that $r(i) = \mathcal{O}(i^{-1-\varepsilon})$ for some $\varepsilon > 0$ almost sure invariance principle holds for Gaussian $(Z_i)_{i=1}^{n}$ with rate $o(n^{1/2})$ (cf. [21]). Then in view of Theorem 2 in [17] a.s. CLT holds. This provides alternative proof of Theorem 2 under a slightly stronger condition.

Let $\big(G(Z_i)\big)_{i=1}^{\infty}$ be a Gaussian subordinated sequence where $G \in \mathcal{L}^2(\mathbb{R}, \phi)$ and $EG(Z_1) = 0$, with $(Z_i)_{i=1}^{\infty}$ being a stationary Gaussian sequence with an arbitrary covariance function. Assume that $\sum_{i=0}^{\infty}\left|r_G(i)\right| < \infty$, where $r_G(i)$ is covariance function of $\big(G(Z_i)\big)_{i=1}^{\infty}$ at lag i. Denote by f a spectral density of the sequence $(Z_i)_{i=1}^{\infty}$ which is assumed to exist. We say that f is logarithmic Lipschitz if

$$\left|f(y) - f(x)\right| \le C\left(\log\frac{1}{|y-x|}\right)^{-\gamma}$$

for $-\pi \le x, y \le \pi$ and some $\gamma > 0$. Obviously, if f is Lipschitz with an exponent γ in the usual sense it is also logarithmic Lipschitz with the same exponent. We prove

Theorem 3. *Assume that f is bounded away from 0 and is logarithmic Lipschitz. If G satisfies the conditions above and $\sigma^2 = r_G(0) + 2\sum_{i=1}^{\infty} r_G(i) \ne 0$ then the sequence $\big(G(Z_i)\big)_{i=1}^{\infty}$ satisfies a.s. CLT.*

Proof. Let $X_i = G(Z_i)$. Since $\sum_{i=1}^{\infty}\left|r_G(i)\right| < \infty$ and $\sigma^2 \ne 0$, $\sigma_n^2/n \to \sigma^2$ and the two conditions on σ_i listed in (2.2) are satisfied. Moreover, Breuer and Major [4] and Giraitis and Surgailis [12] proved that $(S_n - ES_n)/\sigma_n \to N(0,1)$ in distribution. Since (2.2) is satisfied, in view of the proof of Theorem 1 in [22] it is enough to show that $\sum_{i=1}^n \alpha_G(i)/i = \mathcal{O}(\log^{1-\varepsilon} n)$, where $\alpha_G(i)$ is α-mixing coefficient at lag i pertaining to the sequence $(X_i)_{i=1}^\infty$ since it implies (1.3). However, $\alpha_G(i) \le \alpha(i) \le \rho(i)$, where $\alpha(\cdot)$ are and $\rho(\cdot)$ are α-mixing and ρ-mixing coefficients, respectively pertaining to the sequence $(Z_i)_{i=1}^\infty$. Thus it is enough to prove that $\sum_{i=1}^n \rho(i)/i = \mathcal{O}(\log^{1-\varepsilon} n)$. Recall that since $(Z_i)_{i=1}^\infty$ is Gaussian ([14])

$$\rho(n) = \sup \int_{-\pi}^{\pi} e^{in\lambda} P(\lambda) Q(\lambda) f(\lambda)\, d\lambda,$$

where supremum is taken over all trigonometric polynomials of the form $P(\lambda) = \sum_{j\ge 0} a_j e^{ij\lambda}$ and $Q(\lambda) = \sum_{j\ge 0} b_j e^{ij\lambda}$ such that

$$\int_{-\pi}^{\pi} \big|P(\lambda)\big|^2 f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \big|Q(\lambda)\big|^2 f(\lambda)\, d\lambda = 1.$$

In view of Jackson's theorem and since f is continuous there exists a trigonometric polynomial $T_n(\lambda)$ of the rank $2n-2$ such that

$$\sup_{-\pi \le \lambda \le \pi} \big|f(\lambda) - T_n(\lambda)\big| \le C\omega(1/n, f),$$

where $\omega(s,f)$ is an uniform modulus of continuity pertaining to a function f. Obviously, $\int_{-\pi}^{\pi} e^{i\lambda k} T_n(\lambda)\, d\lambda = 0$ for any $k > 2n-2$. Thus for such k

$$\begin{aligned}\rho(k) &= \sup_{P,Q} \left| \int e^{ik\lambda} P(\lambda) Q(\lambda) \big(f(\lambda) - T_n(\lambda)\big)\, d\lambda \right| \\ &\le C\omega(1/n, f) \int_{-\pi}^{\pi} \big|P(\lambda)\big| Q(\lambda) \big(f(\lambda)/m\big)\, d\lambda,\end{aligned}$$

where $m := \inf f > 0$. Using Cauchy inequality we have

$$\begin{aligned}\rho(k) &\le \frac{C}{m}\omega(1/n, f) \left(\int_{-\pi}^{\pi} \big|P(\lambda)\big|^2 f(\lambda)\, d\lambda \right)^{1/2} \left(\int_{-\pi}^{\pi} \big|Q(\lambda)\big|^2 f(\lambda)\, d\lambda \right)^{1/2} \\ &= \frac{C}{m}\omega(1/n, f).\end{aligned}$$

Thus it follows that $\rho(n) \le (C/m)\omega(2/n, f)$ and

$$\sum_{i=1}^n \frac{\rho(i)}{i} \le \frac{C}{m} \sum_{i=1}^n \frac{\omega(2/i, f)}{i} = \mathcal{O}\Big(\sum_{i=1}^n \frac{1}{i(\log i)^\gamma} \Big) = \mathcal{O}(\log^{1-\gamma} n).$$

Thus condition (1.3) is satisfied with $\varepsilon = \gamma$. ■

We consider now a functional version of (1.1) for a short-range dependent subordinated Gaussian process. Namely, define the usual linear approximation to a partial sum process

$$s_n(t,\omega) = \begin{cases} \sigma_n^{-1} S_k(\omega), & \text{if } t = k/n,\ k = 0,1,\dots,n; \\ \text{linear in between,} & \text{otherwise} \end{cases}$$

and denote by $\delta(x)$ the point mass at $x \in \mathcal{C}[0,1]$. Then the functional counterpart of a.s. CLT in (1.1) states that

$$(2.8) \qquad (\log n)^{-1} \sum_{k=1}^{n} \frac{1}{k} \delta\big(s_k(\cdot,\omega)\big) \quad \overset{\mathcal{D}}{\to} \quad W$$

a.s., where $W(\cdot)$ is standard Brownian motion and $\overset{\mathcal{D}}{\to}$ denotes convergence in distribution in $\mathcal{C}[0,1]$. We now prove

Theorem 4. *Assume that the assumptions of Theorem 3 are satisfied and moreover $EG^4(Z_1) < \infty$. Then the functional a.s. CLT (2.8) holds.*

Proof. We have that $s_n(\cdot) \overset{\mathcal{D}}{\to} W(\cdot)$ provided $EG^4(Z_1) < \infty$ (cf. [9]) and this implies $(\log n)^{-1} \sum_{k=1}^{n} k^{-1} Ef\big(s_k(\cdot,\omega)\big) \to f(W)$ for any bounded Lipschitz function $f : \mathcal{C}[0,1] \mapsto R$. Thus an obvious adaptation of the proof of Theorem 1 in [22] indicates that in order to prove (2.8) it is sufficient to prove a functional counterpart of (1.3)

$$(2.9) \qquad \operatorname{Var}\left(\sum_{k=1}^{n} \frac{1}{k} f\big(s_k(\cdot,\omega)\big) \right) = \mathcal{O}(\log^{2-\varepsilon} n),$$

where f is an arbitrary bounded Lipschitz function $f : \mathcal{C}[0,1] \mapsto \mathbb{R}$. As in the proof of Theorem 1 we note that it is enough to bound

$$\sum_{i<j} (ij)^{-1} \operatorname{Cov}\big(f(s_i), f(s_j)\big).$$

Observe that the sum over $\{i < j \le 2i\}$ can be bounded using a standard argument by $C \log n$. Thus it is enough to consider summation over the complement of this set. For $2i < j$ define

$$r_{ij}(t,\omega) = \begin{cases} 0, & \text{if } t \le 2i/j; \\ \dfrac{\big(S_k(\omega) - S_{2i}(\omega)\big)}{\sigma_j} & \text{for } t = k/j \text{ and } k > 2i; \\ \text{linear in between,} & \text{otherwise} \end{cases}$$

and write

$$\operatorname{Cov}\left(f(s_i), f(s_j)\right) = \operatorname{Cov}\left(f(s_i), f(s_j) - f(r_{ij})\right) + \operatorname{Cov}\left(f(s_i), f(r_{ij})\right).$$

Observe that $f(s_i)$ and $f(r_{ij})$ are $\sigma(X_1, \ldots, X_i)$ and $\sigma(X_{2i+1}, \ldots, X_j)$-measurable, respectively. Since f is bounded, Wolkonski–Rozanov inequality (cf. e.g. [10, p. 10]) implies $\left|\operatorname{Cov}\left(f(s_i), f(r_{ij})\right)\right| \le C\alpha_G(i)$ and

$$\sum_{1\le i<j\le n} \frac{\alpha_G(i)}{ij} \le C \sum_{i\le n} \frac{\log i}{i(\log i)^\gamma} = \mathcal{O}(\log^{2-\gamma} n) \tag{2.10}$$

as shown in the proof of Theorem 3. Moreover, since f is bounded and Lipschitz

$$\left|\operatorname{Cov}\left(f(s_i), f(s_j) - f(r_{ij})\right)\right| \le C\|s_j - r_{ij}\|_\infty = CE\Big(\max_{k\le 2i} |S_k|/\sigma_j\Big). \tag{2.11}$$

We use Móricz [20] inequality: if for nonnegative constants a_j, $j = 1, \ldots, n$ and $p > 0, q > 1$, $E\big(|S_i|^p\big) \le \big(\sum_{j=1}^i a_j\big)^q$ for $1 \le i \le n$, then

$$E\Big(\max_{1\le i\le n} |S_i|^p\Big) \le C_{pq}\Big(\sum_{i=1}^n a_j\Big)^q \tag{2.12}$$

Since (2) in [9] states that $ES_i^4 \le C_0 i^2, i = 1, 2, \ldots$ provided $EG^4(Z_1) < \infty$, an application of (2.12) with $p = 4$, $q = 2$ and $a_j = C_0^{1/2}$, $j = 1, \ldots, n$, yields

$$E\Big(\max_{k\le 2i} |S_k|\Big) \le \Big\{E\Big(\max_{k\le 2i} |S_k|^4\Big)\Big\}^{1/4} \le C_{42}^{1/4} C_0^{1/4} (2i)^{1/2}$$

and since $\sigma_j^2 \ge Cj$ the inequality above implies that (2.11) is less than $C(i/j)^{1/2}$. Thus

$$\sum_{\substack{1\le i<j\le n\\ 2i<j}} \frac{1}{ij}\left|\operatorname{Cov}\left(f(s_i), f(s_j) - f(r_{ij})\right)\right| \le C \sum_{1\le i<j\le n} \frac{1}{ij}\Big(\frac{i}{j}\Big)^{1/2} = \mathcal{O}(\log n). \tag{2.13}$$

Equations (2.10) and (2.13) imply (2.9). ■

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